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Property Rights and Efficiency in OLG Models with Endogenous Fertility

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Property Rights and Efficiency in OLG Models with Endogenous Fertility*

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Abstract

Is there an economic rationale for pronatalist policies? We propose and analyze a particular market failure that leads to inefficiently low fertility in equilibrium. The friction is caused by the lack of ownership of children: if parents have no claim on their children's income, the private benefit from producing a child can be smaller than the social benefit. We analyze an overlapping-generations model with fertility choice and parental altruism. Ownership is modeled as a minimum constraint on transfers from parents to children. Using the efficiency concepts proposed in Golosov, Jones, and Tertilt (2007), we find that whenever the transfer floor is binding, fertility choices are inefficient. Second, we show that the usual conditions for efficiency are not sufficient in this context. Third, in contrast to settings with exogenous fertility, a PAYG social security system cannot be used to implement efficient allocations. To achieve an efficient outcome, government transfers need to be tied to fertility choice.

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1 Introduction

In many European countries current birth rates are well below replacement levels, e.g., as low as 1.4 in Germany or 1.3 in Italy. Governments in those countries appear concerned that fertility is "too low", and are discussing several pronatalist policies.[1] To some extent, these policies have already been implemented in various countries. For example, French parents receive generous subsidies for each child. Some Italian villages have experimented with generous one-time payments for the birth of a child.[2]

In this paper we ask in what sense fertility may be "too low" and explore the ensuing economic rationale for pronatalist policies. The friction we investigate is related to ownership over children.[3] The basic observation is that children are a resource for society. In particular, they increase the total labor endowment in the future. Property rights over this resource affect incentives. If labor income belongs to children rather than parents, then the private benefit (to parents) of producing children may be smaller than the social benefit and hence fertility may be inefficiently low.[4]

To understand the main mechanism, consider the following simple example. People live for two periods but are endowed with labor only when young. Suppose people derive utility from consumption but not from children. Further assume that parents have no access to their children's resources. Then, with a positive cost of bearing children, equilibrium fertility will be zero. However, as long as labor is an essential input into production, this means output when those initial people are old will also be zero. That is, old people will be miserable. They would like to have workers around to produce consumption goods and in fact this would be feasible. But there is no incentive for anyone to produce such workers. Instead, assume now that parents have a claim on their children's income. Then, as long as the claim is large enough relative to the cost of bearing children, people will indeed have children. Output in the second period will then be positive and everyone is better off. One could argue that children are no worse off either, since in the former scenario they are not even alive.

[1]See for example "Europe, East and West, wrestles with falling birthrates—Long decline threatening economy," International Herald Tribune (September 3, 2006) and "Europe: The fertility bust, Charlemagne" The Economist, February 11, 2006.

[2]See "European nations offer incentives to have kids", San Francisco Chronicle, Elizabeth Bryant, August 10, 2008; "Where have all the bambini gone?", Telegraph, April 18, 2004.

[3]In this paper we use the words "child" and "offspring" as synonyms.

[4]Other inefficiencies relating to fertility are addressed in Pitchford (1985), Nerlove, Razin, and Sadka (1985, Oct., 1986), Lee and Miller (1990), Bruce and Waldman (1990), Harford (1998), Zhang and Zhang (2007). These papers concentrate on strategic considerations and a variety of externalities such as pollution.

In this example, increasing fertility by shifting property rights from children to parents seems to be a Pareto improvement. However, Pareto efficiency is not well-defined in models with endogenous populations.[5] We therefore use the concepts of $\mathcal{A}-$ and $\mathcal{P}-$efficiency proposed by Golosov, Jones, and Tertilt (2007) which allow for such comparisons. A feasible allocation is $\mathcal{A}-$efficient if there is no other feasible allocation such that no one *alive in both allocations* is worse off and at least one person *alive in both allocations* is strictly better off. The definition of $\mathcal{P}-$efficiency is similar, except that all *potential* people, including the unborn, are considered. In the example, the equilibrium allocation where parents have some property rights over children's income $\mathcal{A}-$dominates the allocation where parents don't have any property rights. With an additional assumption on the utility of not being born, it also $\mathcal{P}-$dominates. Note that the dominating allocation involves more people.

The above example is clearly an extreme one. Most models of fertility choice view children as a consumption good,[6] a utility function in the case of parental altruism[7] or both.[8] This paper argues that the basic problem, namely misaligned property rights leading to inefficiently low fertility, is present also in these more general settings. Since property rights over children vary substantially across countries and have changed dramatically over time (from parents towards children in most countries), analyzing the implications seems important.[9]

The model we use is an infinite horizon overlapping-generations (OLG) model with fertility choice and parental altruism. We formalize the idea of property rights by introducing a constraint that sets a minimal transfer from parents to children. This formulation allows us to cover the full range of possible property rights, from parents fully owning children's labor income (when large negative transfers are allowed) to a situation where children have a legal claim on their parent's income (a positive minimal transfer). When property rights are more tilted towards children, the constraint is more likely to bind.[10]

[5]Of course, one can ask if, holding population size constant, a Pareto-dominating allocation exists. However, such analysis yields no answer to the question whether equilibrium fertility is inefficiently high or low.

[6]e.g. Becker (1960), Eckstein and Wolpin (1985), Conde-Ruiz, Giménez, and Pérez-Nievas (2010), etc.

[7]e.g. Barro (1974), Carmichael (1982), Burbidge (1983), etc.

[8]e.g., Razin and Ben-Zion (1975), Pazner and Razin (1980), Becker and Barro (1986, 1988), Barro and Becker (1989), etc.

[9]We document different laws related to child ownership in Schoonbroodt and Tertilt (2010).

[10]Even though in the formal model we focus on property rights over the labor endowment, our conclusions hold more generally. For example, if parents and children disagree about other aspects of a child's life, then who owns the right to make decisions will affect fertility choices and efficiency.

This paper makes three contributions to the literature. First, we show that when parents do not have enough property rights, equilibrium fertility will be inefficiently low. More specifically, we prove that an equilibrium allocation is $\mathcal{A}-$ and $\mathcal{P}-$efficient if and only if parents are not transfer constrained. Whenever the transfer constraint is binding, we show how an $\mathcal{A}-$ and $\mathcal{P}-$dominating allocation can be constructed that involves more people. Therefore, this inefficiency provides a potential rationale for government intervention aimed to increase fertility.

Second, we revisit the literature on efficiency in OLG models. Our set-up generalizes the basic OLG model along two dimensions: it allows for parental altruism and for endogenous fertility choice. Table 1 classifies the literature along these two dimensions. First, we show that with endogenous fertility and parental altruism, the usual steady-state conditions for efficiency are not sufficient for $\mathcal{A}-$efficiency. For example, the condition for dynamic efficiency, that the interest rate must be above the population growth rate, is not sufficient for $\mathcal{A}-$efficiency. The reason is that in addition to over-accumulation of capital, under-accumulation of people (i.e. labor) can also be a problem. Second, an important dimension that has been neglected in the literature is how the allocation of property rights determines whether equilibrium allocations are efficient. In fact, non-altruistic models assume that every generation owns their labor income, while altruistic models often assume that parameters are such that transfer constraints are not binding (i.e. parents have "enough" property right). We show that it is precisely the combination of property rights and altruism that is important for efficiency. This finding is not specific to $\mathcal{A}-$efficiency. Property rights matter for equilibrium efficiency also when more conventional efficiency concepts are used. In particular, we show how the thresholds of property rights beyond which different types of inefficiencies occur depends on the degree of altruism.

Our third contribution concerns policy implications. We show that, in contrast to OLG models with exogenous fertility, a pay-as-you-go (PAYG) system cannot be used to implement $\mathcal{A}-$efficient allocations. Even if the pension system is such that the transfer constraint is not binding, the resulting equilibrium is typically not $\mathcal{A}-$efficient. The reason is that, when choosing fertility, parents do not take into account that they are also producing future contributors to the pension system. Thus, the costs and benefits of having children are not aligned in a normal PAYG system. An alternative policy—a fertility dependent PAYG system—on the other hand, can be used to implement an $\mathcal{A}-$efficient allocation. Interestingly, we find that the same allocation can also be im-

Table 1: Literature Comparison

	exogenous fertility	endogenous fertility
without altruism	Samuelson (1958), Cass (1972), Balasko and Shell (1980)	Eckstein and Wolpin (1985), Abio, Mahieu, and Patxot (2004), Lang (2005), Michel and Wigniolle (2007, 2009), Conde-Ruiz, Giménez, and Pérez-Nievas (2010)
with altruism	Barro (1974), Burbidge (1983)	Razin and Ben-Zion (1975), Pazner and Razin (1980), This Paper

plemented through birth subsidies financed by government debt and taxes.[11] This provides a potential rationale for currently observed government policies that subsidize children.

The idea that parents' inability to access a child's future income may lead to inefficiencies has been explored in several other contexts. In particular, several models with exogenous fertility look at the importance of this margin for education decisions. What we call property rights assigned to the child, is sometimes called "borrowing constraints" or "incomplete markets" in the literature. For example, Aiyagari, Greenwood, and Seshadri (2002) analyze the implications of borrowing constraints for the efficiency of investments in children in a model where fertility is exogenous. Similarly, Fernández and Rogerson (2001) analyze the implications of borrowing constraints for child schooling decisions and long-run inequality in a set-up with exogenous (but stochastic) fertility.[12] Also, Boldrin and Montes (2005) analyze a model where young adults make their own schooling decisions but are borrowing-constrained leading to an inefficiently low level of schooling. There is an important distinction, however, between the inefficiency in education and fertility choices. The cost and benefits of investing in human capital could, in principle, be borne by the same person. For example, if children made their own education investment decisions and markets were complete, then there would be no friction. The same is not possible in the context of fertility decisions since it is not technologically feasible for a child to bear the costs of producing itself.

The remainder of the paper is organized as follows. In Section 2 we formalize the

[11] For related optimal fertility policies in different setups, see Cigno (1983, 1986, 1992).
[12] See also Lazear (1983).

example above to illustrate the basic friction and how it relates to property rights. Section 3 presents the model and characterizes equilibria. In Section 4 we analyze the efficiency properties of equilibrium fertility. Section 5 explores several government policies and Section 6 concludes.

2 An Example

We start with a simple example to illustrate that when children have full property rights over themselves, fertility may be too low in the sense that increasing fertility makes everyone better off.

Assume that there are two periods and two generations: parents and, potentially, children. There is a continuum of measure one of identical parents who live for two periods. The utility function of a parent is $\ln(c^m) + \beta \ln(c^o)$, where c^m is consumption when middle-aged and c^o is the parent's consumption when old. Each parent is endowed with w^m units of the consumption good when middle-aged. Parents can save for old age, s, and choose fertility, n. It costs θ units of the consumption good to produce a child. Children, if born, are adults in period two, endowed with one unit of labor, and only value own consumption, c^k. The production function in the second period is $Y = K^\alpha L^{1-\alpha}$ and we assume full depreciation of the capital stock. Assuming perfect competition, factors (labor and capital) earn their marginal products. Savings are invested as capital so that market clearing requires $s = K$. Labor market clearing requires $L = n$.

Now suppose parents have no control over their children's actions, more specifically, over their children's income. We label this case as children fully owning themselves. Then, in equilibrium, no individual parent will have an incentive to have children. The reason is that having children is costly, and that they provide no benefit to their parents. Given that no children are born, there is no labor force in period two and hence output is zero as well. The return on savings is zero, and parent's consumption in period two must be zero. Note that, since every parent is infinitesimal, individual fertility choices, n, do not change aggregate labor supply and hence do not affect prices.

On the other hand, if parents had property rights over part of their children's wages, say an amount ω, then there is an incentive to have children. From a parents perspective there are two investment goods. The return to savings is r, while the return to children is $\frac{\omega}{\theta}$. In equilibrium the interest rate will adjust such that the no-arbitrage condition between both investments holds. In this case parents are clearly better off as utility is logarithmic and they have positive consumption in period 2. As long as the children's utility from consuming $w^k - \omega$ is larger than the utility from not being born, children also benefit from parents having property rights over part of their children's income.

The reason why equilibrium fertility may be too low depending on the allocation of property rights is a missing market. There is no market for private contracts between parents and children where children promise to compensate parents for child-bearing expenses. Clearly, unborn people cannot write such contracts with their parents, but once they are born children have no incentive to sign such a contract. Knowing this, parents don't bear the children in the first place. This missing market problem is overcome by assigning parents property rights over part of their children's income.

The above example in which there is no utility benefit from children is clearly an extreme one. Most models of fertility choice view children as a consumption good (e.g. Becker (1960), Eckstein and Wolpin (1985), Conde-Ruiz, Giménez, and Pérez-Nievas (2010)), a utility function in the case of parental altruism (e.g. Barro (1974), Carmichael (1982), Burbidge (1983)) or both (e.g., Razin and Ben-Zion (1975), Pazner and Razin (1980), Becker and Barro (1986, 1988), Barro and Becker (1989)). However, as we will show throughout this paper, the basic problem (misaligned property rights leading to inefficiently low fertility) is very general and not an artifact of this stark example.

Finally, note that even though in this example, shifting property rights from children to parents seems to be a Pareto improvement, Pareto efficiency is not well-defined in models with endogenous fertility. Instead, we use $\mathcal{A}-$ and $\mathcal{P}-$efficiency, first proposed by Golosov, Jones, and Tertilt (2007), for which we will provide formal definitions in Section 4.

3 The Model

We now set up our model of fertility choice with altruistic parents. The model encompasses the dynastic endogenous fertility models first developed in Razin and Ben-Zion (1975) (where utility is separable in number and utility of children) as well as those in Becker and Barro (1986, 1988) and Barro and Becker (1989) (where utility from the number of children and children's utility is multiplicative), though extended to two-periods of adult life. In contrast to the existing literature, we explicitly introduce ownership over children. Specifically, we focus on property rights over adult children's labor income.

First, we characterize equilibria in general. In the next section we derive efficiency results and compare them to those in other OLG models.

3.1 Model Setup

People live for three periods: childhood, (middle-aged) adulthood and retirement. In childhood, no decisions are made. Middle-aged adults work and bear children. Retired people live off their savings and potentially transfers from their children.[13] Households derive utility from their own consumption when middle-aged, c_t^m, and when old, c_{t+1}^o, the number of children, n_t, as well as their offsprings' average utility. That is, in our model children are a consumption good in that n_t directly enters the utility function, but parents are also altruistic and care about their children's utility.

The utility of a middle-aged household in period t (born in $t-1$) is given by:

$$U_t = u(c_t^m) + \beta u(c_{t+1}^o) + V\left(n_t, \frac{\int_0^{n_t} U_{t+1}^i di}{n_t}\right) \tag{1}$$

We assume that $u(\cdot)$ is continuous, strictly increasing, strictly concave and $u'(0) = \infty$. Discounting between periods is given by β. Later, we introduce specific functional forms for V. For now, we assume that V is strictly increasing and strictly concave in fertility, n_t, and weakly increasing in the average utility of children.

The budget constraints are given by

$$\begin{aligned}
&c_t^m + \theta_t n_t + s_{t+1} \leq w_t(1+b_t) \\
&c_{t+1}^o + \int_0^{n_t} b_{t+1}^i w_{t+1} di \leq r_{t+1} s_{t+1} \\
&b_{t+1}^i \geq \underline{b}_{t+1} \\
&c_t^m, c_{t+1}^o, n_t \geq 0
\end{aligned} \tag{2}$$

where s_{t+1} are savings, $b_{t+1}^i w_{t+1}$ is the transfer from parent to child i if positive, from child i to the parent if negative, and θ_t is the cost per child.[14]

[13] We introduce government transfers in Section 5.

[14] For example, $\theta_t = a_t^g + (a_t^c - \kappa_t) w_t$ with a_t^g the goods cost of children, a_t^c is the fraction of time that has to be spent with every child in raising it and κ_t is the amount of (effective) labor the parent can

The minimum constraint, $\underline{b}_{t+1}$, can be interpreted as parental property rights over children's labor income.[15] When $\underline{b}_{t+1}$ is positive, then a larger transfer floor implies that parents have to bequeath more resources to their children. When $\underline{b}_{t+1}$ is negative, a higher transfer floor means parents can expropriate fewer resources from their children. The transfer floor is only well-defined between -1 and some $\underline{b}^{max}$. When $\underline{b}_{t+1} = -1$ then there are no (legal or effective) constraints on transfers and parents have full property rights over their children's income. If, on the other hand, $\underline{b}_{t+1} = 0$ then children own their labor income. If $\underline{b}_{t+1} > 0$ then children have a claim to their parent's income. The maximum possible transfer, $\underline{b}_{t+1}^{max}$, depends on endogenous variables. At $\underline{b}_{t+1}^{max}$ a parent would save his entire income and leave it to his children. A closed form expression for b^{max} as a function of parameters is derived for specific functional forms in Appendix A.3.

Initially, there is a mass 1 of initial old people each endowed with K_0 capital and n_{-1} children. The initial old chooses $(c_0^o, \{b_0^i\}_{i=0}^{n_{-1}})$ to maximize

$$U_{-1} = \beta u(c_0^o) + V\left(n_{-1}, \frac{\int_0^{n_{-1}} U_0^i di}{n_{-1}}\right)$$

subject to:

$$c_0^o + \int_0^{n_{-1}} b_0^i w_0 di \leq r_0 K_0, \quad b_0^i \geq \underline{b}_0$$

The middle-aged adult in period t chooses $(c_t^m, c_{t+1}^o, n_t, s_{t+1}, \{b_{t+1}^i\}_{i=0}^{n_t})$ to maximize U_t in equation (1) subject to the constraints in (2), given b_t, the transfer from his own parents and prices (w_t, w_{t+1}, r_{t+1}), taking the behavior of all descendants as given.

Since $u(.)$ is strictly concave and there is no heterogeneity among children, it is always best for the parent to give the same transfer to each child, $b_{t+1}^i = b_{t+1}, \forall i$. Hence, we can rewrite the utility as:

$$U_t = u(c_t^m) + \beta u(c_{t+1}^o) + V(n_t, U_{t+1}) \tag{3}$$

extract from the child. For example, if a period is 20 years and children can work from age 10 and are half as productive as an adult, then $\kappa_t \approx 0.25$. Below we concentrate on parents' property rights over adult children but a change in κ_t could reflect changes in child-labor laws, for example.

[15]Specifying transfers as absolute amounts rather than proportional to the wage leads to the same qualitative results. This is because, though chosen by the parent, both types of transfers are lump-sum to the child since labor supply in perfectly inelastic.

and the budget constraint when old as:

$$c^o_{t+1} + n_t b_{t+1} w_{t+1} \leq r_{t+1} s_{t+1}$$

As can be seen, the constraint set is not convex in general. This is because n multiplies b in the budget constraint when old and both are choice variables. Therefore, the first-order conditions of this problem, while necessary, are not sufficient for an optimum. Instead of using second-order conditions to characterize the solution, one way to circumvent this problem is to follow Alvarez (1999) and to write the utility and constraints in terms of dynasty aggregates. See Appendix A.1 for details.

The representative firm has a neo-classical production function $Y_t = F(K_t, L_t)$, and takes prices (r_t, w_t) as given when choosing (K_t, L_t) to maximize profits. For simplicity, we assume full depreciation throughout.

Next, markets clear. Labor markets clear in period t if the firm's labor demand per old person, L_t, is equal to the number of middle-aged people per old person, n_{t-1}, since they are the only ones who are productive and labor is supplied inelastically. The capital stock per old person, K_t, must be equal to savings from currently old people, s_t. Hence, factor markets clear if $L_t = n_{t-1}$ and $K_t = s_t$. Goods market clearing in period t, expressed in per old person terms, is:

$$c^o_t + n_{t-1}(c^m_t + \theta_t n_t + s_{t+1}) = Y_t.$$

Finally, feasibility per old person is given by:

$$c^o_t + n_{t-1}(c^m_t + \theta_t n_t + K_{t+1}) = F(K_t, L_t). \tag{4}$$

3.2 Characterizing equilibria

Let V_n and V_U denote the derivative of V with respect to its first and second argument, respectively. Let $\lambda_{b,t+1}$ be the Lagrange multiplier on the transfer constraint, $b^i_{t+1} \geq \underline{b}_{t+1}$, in (2). The first-order conditions for the household problem are

$$\begin{aligned}
u'(c^m_t) &= \beta u'(c^o_{t+1}) r_{t+1}, & (5)\\
V_n(n_t, U_{t+1}) &= u'(c^m_t)\theta_t + \beta u'(c^o_{t+1}) b_{t+1} w_{t+1}, & (6)\\
\beta u'(c^o_{t+1}) n_t &= V_U(n_t, U_{t+1}) u'(c^m_{t+1}) + \frac{\lambda_{b,t+1}}{w_{t+1}}, & (7)
\end{aligned}$$

together with the budget constraints when middle-aged and when old. The first two equations are intertemporal conditions equating marginal costs and benefits of savings and fertility. The third condition is an *intra*temporal but *inter*generational condition, equating the parent's marginal cost and benefit of an additional unit of transfer per child, b_{t+1}, unless the minimum constraint is binding.

Denoting k_t the capital stock per worker, the first-order conditions for the firm's problem are given by

$$w_t = F_L(k_t, 1), \tag{8}$$

$$r_t = F_K(k_t, 1). \tag{9}$$

If $\underline{b}_t = -1, \forall t$ and $V_U > 0$, then $\lambda_{b,t} = 0, \forall t$ and we denote the equilibrium allocation by $\{c_t^{m*}, c_{t+1}^{o*}, n_t^*, s_{t+1}^*, k_t^*, b_{t+1}^*\}_{t=0}^{\infty}$ and prices by $\{w_t^*, r_t^*\}_{t=0}^{\infty}$. We denote any equilibrium allocation for the case where some generation is constrained by $\{\hat{c}_t^m, \hat{c}_{t+1}^o, \hat{n}_t, \hat{s}_{t+1}, \hat{k}_t, \hat{b}_{t+1}\}_{t=0}^{\infty}$ and prices by $\{\hat{w}_t, \hat{r}_t\}_{t=0}^{\infty}$.

The state variable in this economy is the capital-labor ratio, K_{t+1}/n_t. Since capital depreciates fully across generations, parents are free to choose the capital-labor ratio optimally, given their constraints. Therefore, if both $\underline{b}_t$ and θ_t are constant, then the economy is in a steady state as of period 1, i.e., there are no transitional dynamics.

3.3 Utility Specifications

To facilitate a comparison with the literature, we sometimes make functional form assumptions for the utility function. However, these assumptions are not needed for most of our results.

We look at two alternative specifications for V. First, we consider the Razin-Ben-Zion (RB) specification given by:

$$V(n_t, U_{t+1}) = \gamma u(n_t) + \zeta U_{t+1} \tag{10}$$

Second, we consider Barro-Becker type altruism (BB) given by[16]

$$V(n_t, U_{t+1}) = \zeta g(n_t) U_{t+1} \tag{11}$$

[16]Note that, while we assume $BB-$type *altruism*, the model here is an extension of the original BB *model* due to the second period of adult life. We are not the first to consider this extension, however, see for example Zhang and Zhang (2007).

Sequentially substituting utility functions from period s to ∞, we get:

$$
\begin{aligned}
RB \qquad & U_s = \sum_{t=s}^{\infty} \zeta^{t-s} \left[u(c_t^m) + \beta u(c_{t+1}^o) + \gamma u(n_t) \right], \\
BB \qquad & U_s = \sum_{t=s}^{\infty} \zeta^{t-s} g(N_{s,t}^m) \left[u(c_t^m) + \beta u(c_{t+1}^o) \right]
\end{aligned}
\tag{12}
$$

where, $N_{s,t}^m$ is the number of middle age descendants of generation s in period t.

Note that there is a special case in which the BB specification and the RB specification coincide (this requires logarithmic utility and a specific functional form for $g(\cdot)$.[17] In general, however, neither of the two specifications is a special case of the other. The RB utility function is particularly useful when comparing our results to results in non-altruistic models with endogenous fertility: simply let $\zeta \to 0$. The $BB-$utility function is richer in that it allows for complementarity or substitutability between the number and utility of children.[18]

4 Property Rights and Efficiency

In this section we analyze the efficiency properties of equilibria in our model. Analyzing normative questions in models with endogenous fertility requires taking a stand on the appropriate concept of efficiency. The problem is that Pareto efficiency is not a well-defined concept in models with endogenous populations. One might still ask whether a given allocation is Pareto efficient, i.e., whether, holding population size constant, a dominating allocation exists. However, this kind of analysis cannot address the question whether equilibrium fertility is too low. We use alternative concepts, $\mathcal{A}-$ and $\mathcal{P}-$efficiency, first proposed by Golosov, Jones, and Tertilt (2007), which are very close to Pareto efficiency but allow us to compare allocations with different population sizes.[19]

We start this section by defining the concepts. We then prove our main result, namely that equilibrium allocations are $\mathcal{A}-$ and $\mathcal{P}-$efficient if and only if the constraint is not binding. We then provide necessary and sufficient conditions for efficiency and

[17] Details on the necessary utility transformations that lead to this result are available upon request.

[18] See Appendix A.1.1 for details.

[19] In the context of models without altruism, some authors have used an alternative concept, Millian efficiency ($\mathcal{M}-$efficiency), which requires potentially dominating allocations to be symmetric across all people within a given generation. We discuss this concept in Section 4.2.

compare them to the previous literature on efficiency in OLG models. In this context, we demonstrate the importance of the allocation of property rights and its interaction with altruism.

4.1 $\mathcal{A}-$ and $\mathcal{P}-$efficiency of competitive equilibrium allocations

We use the efficiency-concepts suggested in Golosov, Jones, and Tertilt (2007), $\mathcal{A}-$ and $\mathcal{P}-$efficiency. We briefly provide the definitions here and refer the reader to Golosov, Jones, and Tertilt (2007) for details.

Let $\mathcal{P}$ be the set of potential people. An allocation $z = \{z_t^i\}_{(t,i)\in\mathcal{P}}$ is a vector of all goods (including children or people in general) over which person i of generation t's utility is defined, z_t^i, for all potential people. Let A be the set of all possible allocations. Further, let A_t^i be the set of all allocations in which person i of generation t is born. To define $\mathcal{A}-$efficiency, the following assumption is needed:

Assumption 1 *For each $(t, i) \in \mathcal{P}$, there is a well defined, real-valued utility function $U_t^i : A_t^i \to \mathbb{R}$.*

Definition 2 *A feasible allocation $z = \{z_t^i\}_{(t,i)}$ is $\mathcal{A}-$efficient if there is no other feasible allocation $\tilde{z}$ such that*

1. $U_t^i(\tilde{z}) \geq U_t^i(z) \;\; \forall (t, i)$ *alive in both allocations;*
2. $U_t^i(\tilde{z}) > U_t^i(z)$ *for some (t, i) alive in both allocations.*

$\mathcal{A}-$efficiency is a natural extension of Pareto efficiency to environments in which the number of people is endogenous. It also has the advantage of not requiring people who are not alive to have preferences. What the concept does is a pairwise comparison of allocations with a focus only on those people who are alive. If someone isn't born in a particular allocation, this person has no "say" in the utility comparison. Alternatively, if one is willing to define utility even for people who are not alive, then another logical extension of Pareto efficiency is a concept where every potential person gets a "say," which is termed $\mathcal{P}-$efficiency by Golosov, Jones, and Tertilt (2007). To define $\mathcal{P}-$efficiency, the following assumption is needed:

Assumption 3 *For each $(t, i) \in \mathcal{P}$, there is a well defined, real-valued utility function $U_t^i : A \to \mathbb{R}$.*

Definition 4 *A feasible allocation $z = \{z_t^i\}_{(t,i)}$ is $\mathcal{P}-$efficient if there is no other feasible allocation $\tilde{z}$ such that*

1. *$U_t^i(\tilde{z}) \geq U_t^i(z)$ for all $(t,i) \in \mathcal{P}$;*

2. *$U_t^i(\tilde{z}) > U_t^i(z))$ for at least one $(t,i) \in \mathcal{P}$.*

Throughout the paper whenever we talk about $\mathcal{P}-$efficiency, we also assume that being alive is always preferred to not being born. For all other concepts, this assumption is irrelevant.

Assumption 5 *$U_t^i(\tilde{z}) < U_t^i(z)$ for all $\tilde{z}$ in which (t,i) is not born and z in which (t,i) is born.*

As shown in Golosov, Jones, and Tertilt (2007), under relatively mild assumptions, the set of $\mathcal{A}-$efficient allocations is a subset of the set of $\mathcal{P}-$efficient allocations. The reason is that there are more ways of $\mathcal{A}-$dominating an allocation because it is allowed to "ignore" people. For many applications, especially in our context here, the two concepts give the same result.

Our first result states that equilibria in an economy without binding transfer constraints are always efficient. Recall that $\lambda_{b,t}$ denotes the multiplier on the transfer constraint $b_{t+1} \geq \underline{b}_{t+1}$.

Proposition 6 *If parameters are such that $\lambda_{b,t} = 0$ for all t, then the equilibrium allocation, $z^* \equiv \{c_t^{m*}, c_{t+1}^{o*}, n_t^*, s_{t+1}^*, k_t^*, b_{t+1}^*\}_{t=0}^{\infty}$, is $\mathcal{A}-$ and $\mathcal{P}-$efficient.*

Proof. This essentially follows from Theorem 2 in Golosov, Jones, and Tertilt (2007). Without binding transfer constraints, the equilibrium allocation maximizes the utility of the dynastic head, given prices. Thus, the equilibrium allocation is dynastically $\mathcal{A}-$ and $\mathcal{P}-$efficient. This, together with the assumption of a neoclassical production function, ensures that the assumptions of Theorem 2 in Golosov, Jones, and Tertilt (2007) are satisfied. ∎

On the other hand, when there are binding constraints, then the equilibrium allocation is essentially always $\mathcal{A}-$ and $\mathcal{P}-$inefficient. The only exception to this result is when parents are not altruistic at all. If $V_U = 0$, then a binding constraint does not necessarily imply inefficiency. We will come back to this special case later. For now, we assume that $V_U > 0$.

Proposition 7 *Assume $V_U > 0$. If parameters are such that $\lambda_{b,s+1} > 0$ for some generation s, then the equilibrium allocation, $\hat{z} \equiv \{\hat{c}_t^m, \hat{c}_{t+1}^o, \hat{n}_t, \hat{s}_{t+1}, \hat{k}_t, \hat{b}_{t+1}\}_{t=0}^{\infty}$, is $\mathcal{A}-$ and $\mathcal{P}-$inefficient.*

Proof. Consider the following alternative allocation, $\tilde{z}$. All the people alive in $\hat{z}$, except individuals of generation s, receive the same as in the equilibrium allocation. That is $\forall t \neq s$:

$$\begin{aligned} \tilde{c}_t^m &= \hat{c}_t^m & \tilde{n}_t &= \hat{n}_t \\ \tilde{c}_{t+1}^o &= \hat{c}_{t+1}^o & \tilde{s}_{t+1} &= \hat{s}_{t+1} \end{aligned}$$

The allocation is different for the individuals from generation s alive in $\hat{z}$. They have ϵ more children, and receive an additional transfer Δ from each new child. More formally, we have

$$\begin{aligned} \tilde{c}_s^m &= \hat{c}_s^m - \theta_s \epsilon & \tilde{n}_s &= \hat{n}_s + \epsilon \\ \tilde{c}_{s+1}^o &= \hat{c}_{s+1}^o + (\Delta - \underline{b}_{s+1}\hat{w}_{s+1})\epsilon & \tilde{s}_{s+1} &= \hat{s}_{s+1}. \end{aligned}$$

That is, they have ϵ more children than in the equilibrium allocation. This ϵ-mass of *new* people (not alive in $\hat{z}$), receive:

$$\tilde{c}_{s+1}^{m,n} = \frac{F(\hat{s}_{s+1}, \tilde{n}_s) - F(\hat{s}_{s+1}, \hat{n}_s)}{\epsilon} - \hat{s}_{s+2} - \theta_{s+1}\hat{n}_{s+1} + \underline{b} - \Delta$$
$$\tilde{c}_{s+1}^{o,n} = \hat{c}_{s+1}^o \qquad \tilde{n}_{s+1}^n = \hat{n}_{s+1} \qquad \tilde{s}_{s+2}^n = \hat{s}_{s+2}$$

That is, the additional people get an equal fraction of the extra output they produce and they give $\left(\Delta - \underline{b}_{s+1}\hat{w}_{s+1}\right)$ each to their parents in period $s+1$—that is, they give Δ more to their parents than their siblings. Note that $V_U > 0$ together with strict concavity of $u(c)$ guarantees that $\hat{c}_{s+1} > 0$ which assures that $\Delta > 0$ is possible. The additional people do, however, have the same fertility, savings, and consumption when old as their siblings. Since production is expressed in per old person terms, we give the descendants of the $\epsilon-$mass of new people the same allocation as other individuals in their generation.

First, note that feasibility (equation 4) of the alternative allocation is satisfied by construction.

Second, we show that, for small ϵ and Δ, the alternative allocation is $\mathcal{A}-$superior to the equilibrium allocation. To do this, for people alive in $\hat{z}$, define $\tilde{U}_t$ to be the utility of generation t under the new allocation and $\hat{U}_t$ under the equilibrium allocation, respectively. Then, it is easy to see that $\tilde{U}_t = \hat{U}_t$ for all $t > s$. Further, for the ϵ-mass of new people, we have:

$$\tilde{U}^n_{s+1}(\epsilon, \Delta) = \hat{U}_{s+1} - u(\hat{c}^m_{s+1}) + u(\tilde{c}^{m,n}_{s+1})$$

For generation s, we have:

$$\tilde{U}_s(\epsilon, \Delta) = u(\tilde{c}^m_s) + \beta u(\tilde{c}^o_{s+1}) + V\left(\tilde{n}_s, \frac{\hat{n}_s \tilde{U}_{s+1} + \epsilon \tilde{U}^n_{s+1}(\epsilon, \Delta)}{\hat{n}_s + \epsilon}\right)$$

Using the the definition of $\tilde{z}$, this is equal to

$$\tilde{U}_s(\epsilon, \Delta) = u(\tilde{c}^m_s) + \beta u(\tilde{c}^o_{s+1}) + V\left(\tilde{n}_s, \frac{\epsilon[u(\tilde{c}^{m,n}_{s+1}) - u(\hat{c}^m_{s+1})]}{\hat{n}_s + \epsilon} + \hat{U}_{s+1}\right)$$

Taking the derivative with respect to ϵ and evaluating the expression at $\epsilon = 0$, we have

$$\begin{aligned}\frac{\partial \tilde{U}_s(\epsilon, \Delta)}{\partial \epsilon}\Big|_{\epsilon=0} = & -\theta u'(\hat{c}^m_s) + \beta u'(\hat{c}^o_{s+1})[\Delta - \underline{b}_{s+1}\hat{w}_{s+1}] \\ & + V_n\left(\hat{n}_s, \hat{U}_{s+1}\right) + V_U\left(\hat{n}_s, \hat{U}_{s+1}\right)\frac{[u(\hat{c}^m_{s+1} - \Delta) - u(\hat{c}^m_{s+1})]}{\hat{n}_s}\end{aligned}$$

Using equation (6), this reduces to

$$\frac{\partial \tilde{U}_s(\epsilon, \Delta)}{\partial \epsilon}\Big|_{\epsilon=0} = \beta u'(\hat{c}^o_{s+1})\Delta + V_U\left(\hat{n}_s, \hat{U}_{s+1}\right)\frac{[u(\hat{c}^m_{s+1} - \Delta) - u(\hat{c}^m_{s+1})]}{\hat{n}_s}$$

Note that for $\Delta = 0$, this expression is zero. So all that is left to show is that for a small increase in Δ, the expression increases. Taking derivatives with respect to Δ and evaluating at $\Delta = 0$, we have:

$$\frac{\frac{\partial \tilde{U}_s(\epsilon,\Delta)}{\partial \epsilon}\big|_{\epsilon=0}}{\partial \Delta}\Big|_{\Delta=0} = \beta u'(\hat{c}^o_{s+1}) - V_U\left(\hat{n}_s, \hat{U}_{s+1}\right)\frac{u'(\hat{c}^m_{s+1})}{\hat{n}_s}$$

By the first order condition (7) this is equal to $\frac{\lambda_{b,s+1}}{\hat{w}_{s+1}\hat{n}_s}$, which is strictly positive if and only if the constraint is binding. Hence, for small positive ϵ and Δ, generation s is strictly better off with the alternative allocation. Finally, any generation t prior to s ($t < s$) has generation s as a descendant and, since $V_U > 0$, is also strictly better off. This completes the proof that the alternative allocation $\mathcal{A}-$dominates the equilibrium allocation.

By Assumption 5, the new people are also strictly better off, and hence the alternative allocation also $\mathcal{P}-$dominates the equilibrium allocation. ■

It is worth noting that the unconstrained equilibrium allocation, though $\mathcal{A}-$efficient, is not necessarily $\mathcal{A}-$superior to the equilibrium allocation when the constraint is binding. This is because, apart from the initial old, every subsequent generation may be worse off when the constraint is removed.

Propositions 6 and 7 are an interesting instance in which Coase's theorem does not apply: the allocation of property rights matters for the efficiency of the equilibrium allocation. The reason is a missing market. Essentially, the market for private contracts between parents and children where children promise to compensate parents for child-bearing expenses does not exist. Clearly, unborn people cannot write such contracts with their parents, but once they are born children have no incentive to sign such a contract. But without such a contract, parents have a reduced incentive to bear children.

Finally, we would like to emphasize that having people truly overlap in their lives is a crucial ingredient for generating inefficiencies. In many fertility models with altruism (e.g. Barro and Becker (1989)) people consume during one period only and hence do not overlap as adults. In our set-up, this corresponds to the special case $\beta = 0$. For this case, equilibrium bequests are always strictly positive. If they weren't, the capital stock would be zero and the interest rate infinite. As long as $V_U > 0$, zero bequests cannot be an optimal choice. For this special case then, any minimum bequest constraint less than or equal to zero (i.e. $\underline{b} \leq 0$) will never be binding and therefore no inefficiencies occur. In our more general set-up (with $\beta > 0$), parents overlap with productive children and therefore desired transfers may well be negative. The difference is that when generations overlap, negative bequests are perfectly consistent with a positive capital stock.[20]

4.2 Necessary and Sufficient Conditions for Efficiency

We now derive necessary and sufficient conditions for efficiency and compare them to the literature in Table 1.

[20] Razin and Ben-Zion (1975) and Pazner and Razin (1980) allow for $\beta > 0$. However, they implicitly assume that $\underline{b}_t = -1$ for all t throughout their analysis.

4.2.1 Comparison with exogenous fertility models: interest and fertility rates

In standard OLG models (top left of Table 1) first developed by Samuelson (1958) and Diamond (1965), the stationary equilibrium allocation is dynamically efficient if and only if $r > n$. This result dates back to Phelps (1965) and Diamond (1965).[21] Adding altruism (bottom left of Table 1), the condition $r > n$ is still necessary and sufficient for Pareto efficiency (see Barro (1974) for $n = 1$ and Burbidge (1983) for $n \neq 1$). Note that the case where fertility is given exogenously is a special case of our model. In our model, $u(n)$ (for the RB specification) or $g(n)$ (for the BB specification) are simply additive or multiplicative constants in utility, while wages net of child costs correspond to wages or endowments in the standard model. Holding fertility fixed, $r > n$ is also a necessary and sufficient condition for Pareto efficiency in our set-up. To see this, let us first define Pareto efficiency for completeness:

Definition 8 *A feasible allocation $z = \{z_t^i\}_{(t,i)}$ is Pareto efficient if there is no other feasible allocation $\tilde{z}$ with the same set of people alive such that*

1. $U_t^i(\tilde{z}) \geq U_t^i(z) \ \ \forall (t,i)$;
2. $U_t^i(\tilde{z}) > U_t^i(z)$ *for some* (t,i).

Lemma 9 *Assume RB or BB. A stationary equilibrium allocation is Pareto efficient if and only if $r > n$.*

Proof. See Appendix A.2.1. ■

However, when fertility is allowed to change, then the condition needs to be modified as follows.

Proposition 10 *Assume RB or BB with $\zeta > 0$. A necessary and sufficient condition for a stationary equilibrium allocation to be $\mathcal{A}-$ (and $\mathcal{P}-$)efficient is*

$$\begin{array}{ll} RB & n = \zeta r \\ BB & n^{\varepsilon} = \zeta r \end{array} \tag{13}$$

Proof. See Appendix A.2.2. ■

The condition is essentially a no-arbitrage condition between investing in savings versus bequests. In equilibrium, the return to investing in savings is r, while the return

[21] See also Cass (1972) and Balasko and Shell (1980).

to bequests depends on the utility function. Each additional unit of bequests is divided by n children, so that—at least in the RB formulation—the return on bequests is equal to ζ/n.

Recall that $\zeta < 1$ is necessary for the model to be well-defined. Thus, Proposition 10 immediately implies that any $\mathcal{A}-$efficient equilibrium allocation was characterized by $r > n$ in RB. More generally, suppose an unconstrained equilibrium is characterized by $r < n$. Then there exists a Pareto dominating allocation where population is held fixed. Of course, the same allocation would also be $\mathcal{A}-$dominating. This would contradict Proposition 6. However, while necessary, the condition $r > n$ is not sufficient for $\mathcal{A}-$efficiency as the next proposition shows.

Proposition 11 *Assume $V_U > 0$. In a stationary equilibrium, $r > n$ is a necessary but not sufficient condition for $\mathcal{A}-$efficiency.*

Proof. See Appendix A.2.3. ■

The result that $r > n$ is not a sufficient condition for $\mathcal{A}-$efficiency may have important implications. Sometimes the $r > n$ criterion is used to assess whether a particular country is dynamically efficient (e.g., Abel, Mankiw, Summers, and Zeckhauser (1989)). This can be relevant in the context of designing social security systems, for example. Our findings suggest that such analysis may have been based on the wrong criterion—given that, by and large, people do choose birth rates in reality.

4.2.2 Comparison with models without altruism: wages and interest rates

Several authors have analyzed models with endogenous fertility but without altruism (see top right of Table 1).

Without altruism, parents do not value their children's consumption and hence the transfer constraint is *always* binding. As long as the legal constraint $\underline{b}$ is not at the feasible minimum, this means that such an equilibrium is not $\mathcal{A}-$efficient. The logic is the same as in the proof of Proposition 7. The logic breaks down if the legal constraint coincides with the feasible minimum, $\underline{b} = -1$. For this special case, the equilibrium is both $\mathcal{A}-$efficient and the constraint is binding. Note, however, that this is a degenerate equilibrium: the initial old expropriate all income from their children, who consequently consume zero, and no children are born. Clearly, the only stationary equilibrium for this case is trivial: no one is alive. We summarize these results in the next proposition.

Proposition 12 *Assume $V_U = 0$. Then the transfer constraint is always binding. There are two cases:*

a) if $\underline{b} > -1$, then the equilibrium is $\mathcal{A}-$ (and $\mathcal{P}-$) inefficient;

b) if $\underline{b} = -1$, then the equilibrium is such that $c_t^m = c_{t+1}^o = n_{t-1} = 0$ for all $t \geq 1$, and the equilibrium is $\mathcal{A}-$ (and $\mathcal{P}-$) efficient.

This proposition shows that non-degenerate equilibria can never be $\mathcal{A}-$efficient when parents are not altruistic. Papers without altruism therefore typically use a different efficiency concept: $\mathcal{M}-$efficiency, which is similar to $\mathcal{A}-$efficiency but requires people within the same generation to be treated symmetrically (i.e. people with the same preferences and endowment get the same consumption-fertility bundle).

Definition 13 *A feasible symmetric allocation $z = \{z_t\}_t$ is $\mathcal{M}-$efficient if there is no other feasible symmetric allocation $\tilde{z}$ such that*

1. $U_t(\tilde{z}) \geq U_t(z) \ \ \forall t$;

2. $U_t(\tilde{z}) > U_t(z)$ for some t.

As shown in Section 3.1, equilibrium allocations are always symmetric in this model. Hence, using $\mathcal{M}-$efficiency in this environment makes sense.

Note also that the set of symmetric $\mathcal{A}-$efficient allocations is a subset of the set of $\mathcal{M}-$efficient allocations. In particular, in our proof of Proposition 7, we constructed a superior allocation that treated new people differently from those who are alive under both allocations. This would not be a $\mathcal{M}-$dominating allocation. In other words, by widening the set of potentially dominating allocations, one can identify inefficiencies that cannot be addressed if symmetry is imposed.

Authors using models without altruism and $\mathcal{M}-$efficiency also find that $r > n$ is not sufficient for $\mathcal{M}-$efficiency. Instead, they find that a sufficient condition for $\mathcal{M}-$efficiency is given by $r\theta > w$ (see Conde-Ruiz, Giménez, and Pérez-Nievas (2010), Proposition 5 and Corollary 2, and Michel and Wigniolle (2007), Proposition 4). Again, we find that $r\theta > w$ is necessary, but not sufficient for $\mathcal{A}-$efficiency.

Proposition 14 *Assume $V_U > 0$. In a stationary equilibrium, $r\theta > w$ is a necessary but not sufficient condition for $\mathcal{A}-$efficiency.*

Proof. See Appendix A.2.4. ∎

At first it seems intuitive that the equation $r\theta = w$ should hold with equality in an unconstrained equilibrium: the cost of children θ needs to equal their discounted benefit w/r. However, the total benefit from children is higher than their monetary return, as they also provide a utility benefit to their parents. Therefore, if $r\theta = w$ held in equilibrium, parents would find it more beneficial to have more children and save less. This would drive down wages and increase the interest rate.[22] The reason $r\theta > w$ is not sufficient is that it does not guarantee that parents are not constrained. As we have shown in Proposition 7, a binding constraint always implies $\mathcal{A}-$inefficiency.

4.3 Property Rights and Efficiency: an Illustration

So far we have derived several necessary and sufficient conditions for a steady state equilibrium to be efficient according to various efficiency criteria. In this section we now illustrate the importance of property rights for equilibrium (in)efficiency. The illustration shows how the tightness of the transfer constraint (higher $\underline{b}$) is related to the types of inefficiencies that occur. Figure 1 shows a stylized description of how steady state interest, wage and fertility rates change as a function of $\underline{b}$. The picture shows four cases separated by three cut-offs in $\underline{b}$. The first cut-off is b^*, the equilibrium bequest for the unconstrained case. The second cut-off, b_M, is the $\underline{b}$ that leads to $w = \theta r$ in equilibrium. The last cut-off, b_P is the $\underline{b}$ such that in equilibrium $r = n$ holds. While the picture is based on a particular computed example, the characterization is fairly general.[23]

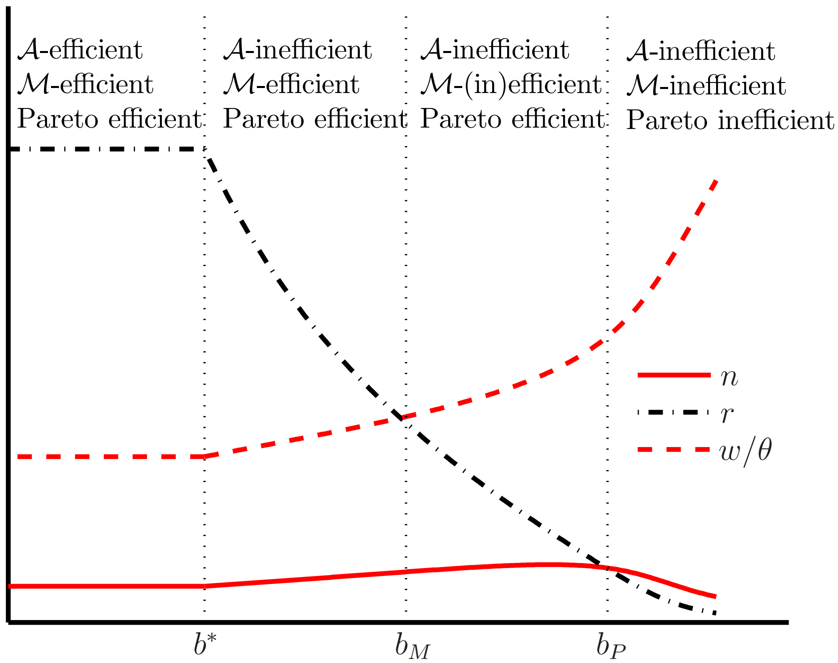
First, for minimum transfers below b^*, the constraint is not binding. This is because with altruism, parents want their children to consume something. In this case, equilibria are $\mathcal{A}-$efficient. This is the result in Proposition 6. We know from Golosov, Jones, and Tertilt (2007) that $\mathcal{A}-$efficiency implies Pareto-efficiency (when fertility is held constant), i.e. the allocation is dynamically efficient.

Second, for $\underline{b}$ above b^* the constraint is binding and the equilibrium allocation is $\mathcal{A}-$inefficient by Proposition 7. When the constraint starts to bind, all else equal, chil-

[22] Only in the special case where the direct utility benefit and indirect utility cost of population cancel out, total population and aggregate capital are pure investment goods to produce aggregate consumption, and the rate of return condition requires $r\theta = w$ at any $\mathcal{A}-$efficient allocation. See Appendix A.1 and A.2.4 for details.

[23] For the RB utility specification with log utility and Cobb-Douglas production closed form solutions for b^*, b_P and b_M exist (see Appendix A.3).

Figure 1: Steady State Characterization as a Function of $\underline{b}$



dren become more expensive. Therefore, parents shift their resources away from children towards savings so that in equilibrium returns to investing in children and in capital are again equalized. This increases the capital-labor ratio causing the interest rate to fall and the wage rate to increase. This allocation is $\mathcal{A}-$inefficient, since an $\mathcal{A}-$planner would choose a lower capital-labor ratio.

As $\underline{b}$ increases, in the example, w monotonically increases and r monotonically decreases in $\underline{b}$.[24] Therefore, as property rights shift even more towards children ($\underline{b}$ increases further), eventually $w \geq r\theta$ holds. Since $w < r\theta$ is a sufficient condition for $\mathcal{M}-$efficiency, it follows that for intermediate values of property rights, the equilibrium allocation is $\mathcal{M}-$efficient but not $\mathcal{A}-$efficient (see Proposition 14). In other words, between b^* and b_M it is possible to dominate the equilibrium allocation only by changing the number of people *and* treating people within the same generation differentially. Beyond b_M it may be possible to dominate an allocation that does not involve asymmetries within the same generation.

[24]The exact nature of the relationship between marginal products and $\underline{b}$ depends on the production and utility functions. Enough substitutability between n and K guarantees that steady state w and r are monotone in $\underline{b}$.

With an even higher minimum transfer constraint, at some point the interest and fertility rates cross. As soon as $r < n$, the allocation becomes Pareto inefficient. If parents are constrained enough, then equilibria are neither $\mathcal{A}-$ nor dynamically efficient. That is, if rights are heavily in favor of children such that $\underline{b} > b_P$, then there exists a dominating allocation that does not involve changing the number of people. In this case, people are saving too much. They are not just picking the wrong portfolio mix (capital vs. children), but the overall level of savings is too high. A dominating allocation can be constructed by redistributing resources across generations (holding population size fixed).

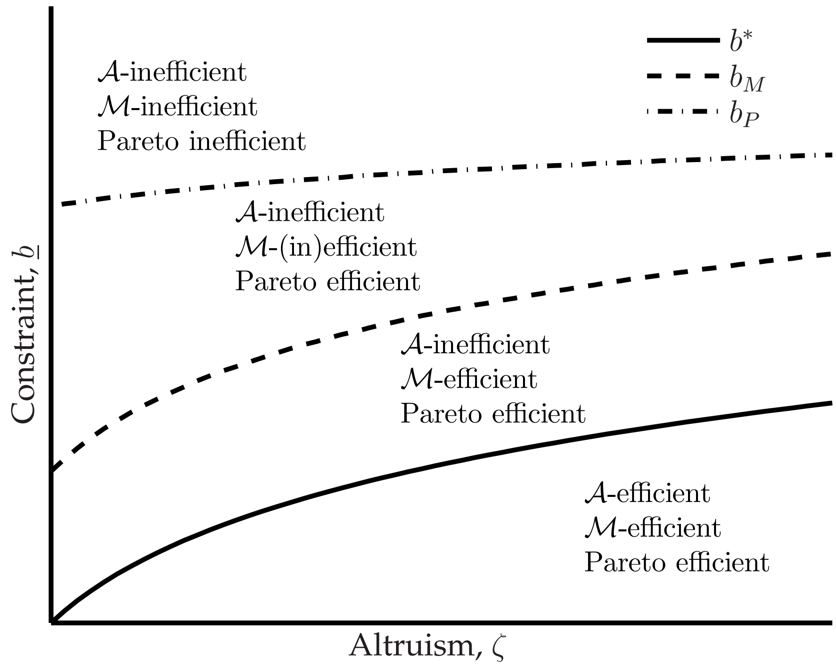
4.4 Property Rights vs. Altruism

There is a strong relationship between the assumption on altruism and (implicit) assumptions on property rights that have been made in the literature. Models without altruism (with or without endogenous fertility) typically assume that $\underline{b} = 0$. On the other hand, authors who use altruistic models typically assume that parents have full property rights. They do this by either assuming that parameters are such that equilibrium bequests are positive, or they assume two sided altruism defined such that all agents alive agree on the appropriate allocation and make intergenerational transfers accordingly. Both of these assumptions are isomorphic to assuming that $\underline{b} = -1$ with one-sided altruism.

In both, the endogenous and exogenous fertility literature, the lack (or misspecification) of altruism has been blamed for inefficiencies occurring in equilibrium.[25] Proposition 7 shows that altruism is perfectly consistent with inefficiencies occurring in equilibrium. In other words, it is not the presence or absence of altruism alone that is the dividing line between equilibrium efficiency and inefficiency. Rather, inefficiencies occur precisely when parents have too few property rights *relative* to their degree of

[25] See for example, Barro (1974). Also, Burbidge (1983) showed that when two-sided altruism is properly added to the standard OLG model, then the interest rate will *always* be larger than the population growth rate, and hence the equilibrium allocation will always be Pareto efficient. This result is derived in the endogenous fertility context by Pazner and Razin (1980), who also find that equilibrium allocations are always dynamically efficient in the sense that $r > n$. Pazner and Razin (1980) is the only previous paper that has used the expression "property rights" in this context. However, they analyze only the case where parents have full property rights. There was a heated debate about these issues at the end of the 1970's and early 1980's. See for example, Drazen (1978), Carmichael (1982), Buiter and Carmichael (1984), Burbidge (1984), Abel (1987) and Laitner (1988). Moreover, Cigno and Werding (2007) (p.121 and p.125) attribute inefficiencies pointed out in Conde-Ruiz, Giménez, and Pérez-Nievas (2010) and Michel and Wigniolle (2007, 2009) to the absence of altruism.

Figure 2: The Interaction between Altruism and Property Rights



altruism. Figure 2 illustrates this point. If altruism is high, then assigning property rights largely to children still leads to equilibrium efficiency. On the other hand, when altruism is low, then parental property rights are crucial for efficiency.

There are several reasons why assigning parents full property rights in altruistic models while assigning children full rights in non-altruistic models is so prevalent in the literature. In models with exogenous fertility and no altruism, parent-child relationships are not even clearly defined and hence the natural starting point is self-ownership for each agent in the economy. Once fertility choice is added there are well-defined family relationships. However, as long as altruism is absent, parents will always take everything they legally or feasibly can from their children. Thus, as shown in Proposition 12, not imposing any transfer constraints implies that only parents consume anything, children starve and the economy ends thereafter—not a very interesting case. Hence, models without altruism typically assume $\underline{b} = 0$. Models with altruism, on the other hand, typically abstract from transfer constraints. This might be partly due to models without constraints being easier to analyze. Also, once altruism is introduced it might appear natural to let a dynastic head make all the decisions for the dynasty.

In sum, Figure 2 shows that it is the combination of property rights and the degree of altruism that determines whether equilibria are efficient or not.

5 Policy Implications

Given the equilibrium inefficiencies resulting from binding transfer constraints, the most obvious policy recommendation would be to simply lift the constraints and give parents full property rights over their children. However, such a policy might not be desirable for various reasons, for example, it might open the door to child abuse. Also, it might be very difficult to enforce payments from adult children to their parents. While these additional concerns are outside of our model, we believe it is useful to explore to what extent alternative policies can also implement efficient allocations in equilibrium.

For example, a pay-as-you-go (PAYG) pension system essentially provides a way of transferring resources from the young to the old. Hence, a PAYG system may be desirable in societies where children have rights over their labor income. In fact, it has been shown that a standard PAYG system can be used to implement Pareto efficient allocations in OLG models with exogenous fertility. However, as we show below, the same logic does not hold in an endogenous fertility set-up. The reason is that a PAYG system may distort the incentives to have children. Therefore, we also examine a fertility dependent PAYG pension system and fertility subsidies financed with government debt. In each case, we ask whether a given policy allows the implementation of $\mathcal{A}-$efficient allocations.

5.1 PAYG social security

We introduce a pay-as-you-go social security system (PAYG) into the model laid out in Section 3. First, we show that the introduction of a standard PAYG social security system, in which children are taxed to finance lump-sum transfers to parents when old increases the desired transfer when parents are constrained, so that for a high enough tax, the bequest constraint is no longer binding. However, such a PAYG system cannot be used to implement an $\mathcal{A}-$efficient allocation. That is, even without a binding constraint, fertility might be inefficiently low in the presence of a PAYG social security system. The reason is that when parents make fertility decisions they do not take into account that they are increasing the number of contributors to the pension system and thereby implicitly their old age support.

To introduce a PAYG system, we make the following modifications to our set-up. The government now taxes middle aged people at rate τ_t and gives the proceeds as a lump-sum pension, T_t, to the old. Both the children and parents take these taxes and pensions as given. Hence, the modified budget constraints are:

$$c_t^m + \theta_t n_t + s_{t+1} \leq w_t(1 + b_t - \tau_t) \tag{14}$$

$$c_{t+1}^o + b_{t+1} w_{t+1} n_t \leq r_{t+1} s_{t+1} + T_{t+1} \tag{15}$$

To simplify algebra, we specify taxes proportional to wages. Note, however, that labor is supplied inelastically, and therefore our specification is equivalent to lump-sum taxes for generation t.

A PAYG system requires the government to balance its budget every period. Hence, in per old person terms, we have $T_{t+1} = n_t \tau_{t+1} w_{t+1}$. That is, the government chooses one instrument, say τ_{t+1}, while the other, T_{t+1}, is determined in equilibrium by the fertility choice of all parents. The (infinitesimal) individual parent realizes that his/her fertility choice alone will not affect the average pension and hence takes T_{t+1} as given. Otherwise, everything in this set-up is the same as before. In particular, other than the budget constraints none of the first-order conditions of the household or the firm and none of the feasibility conditions are affected by this change.

First, assume that $\underline{b}$ is high enough so that the transfer constraint is binding. Then the equilibrium allocation is inefficient. The proof proceeds along the same lines as the proof of Proposition 7 and is hence omitted.

If τ is high enough, then the transfer constraint ceases to bind. To see this, recall that if $\lambda_{s+1} > 0$, from equation (7) we have

$$\beta u'(c_{t+1}^o) n_t > V_U(n_t, U_{t+1}) u'(c_{t+1}^m).$$

Ceteris Paribus, the introduction of a PAYG pension system increases c_{t+1}^o and decreases c_{t+1}^m, which increases the right hand side and decreases the left hand side of the inequality. Thus, for a large enough tax system the transfer constraint ceases to bind. For example, if $\tau_{t+1} = (1 + \underline{b}_{t+1})$, the government takes all income (including legal transfers from parents) away from children. Therefore, the parent would actually want to give more than the legal minimum, $b_{t+1} > \underline{b}_{t+1}$, to assure that the child's consumption is positive.

Even though transfers can be operative if the PAYG tax is large enough (i.e., the constraint may be irrelevant), the resulting equilibrium is nevertheless $\mathcal{A}-$inefficient. A PAYG system leads to underprovision of children because the societal benefit from more children (namely a larger pension payment) is not taken into account when parents make fertility choices. To see this, combine the budget constraints in equations (14) and (15) to get

$$c_{t+1}^o + n_t(c_{t+1}^m + \theta_t n_{t+1} + s_{t+2} - w_{t+1} + \tau_{t+1} w_{t+1}) \leq r_{t+1} s_{t+1} + T_{t+1}$$

It is immediately apparent that the "lump-sum" tax on children, τ_{t+1}, is distortionary to the parent: the more children he/she has, the more taxes his/her dynasty pays. That is, parents do not internalize that children are future contributors to the social security system, T_{t+1}, and therefore do not produce the efficient number of children.[26] Formally, we have:

Proposition 15 *Any equilibrium allocation, z, with a PAYG system is $\mathcal{A}-$inefficient.*

Proof. The proof when the transfer constraint is binding is very similar to the proof of Proposition 7 and hence omitted. The case of the non-binding constraint is more surprising. Such an equilibrium allocation can be $\mathcal{A}-$dominated as follows. Consider some generation s and add ϵ mass of children to this generation. Specifically, consider an alternative allocation $\tilde{z}$ defined as follows: $\tilde{n}_s = n_s + \epsilon$, $\tilde{c}_s^m = c_s^m - \epsilon\theta$, $\tilde{c}_s^o = c_s^o + (\tau - b_{s+1}) w_{s+1}\epsilon$. To assure feasibility, the additional ϵ mass of newborn children consume the following when middle aged:

$$\tilde{c}_n^m = \frac{F(s_{s+1}, \tilde{n}_s) - F(s_{s+1}, n_s)}{\epsilon} - s_{s+2} - \theta_{s+1} n_{s+1} - (\tau - b_{s+1})$$

Everyone else (any generation other than s and also the original children of generation s) consume exactly the same as in the original allocation. It is left to show that the life-time utility for generation s increases in ϵ for small ϵ.

The utility function of generation s as a function of ϵ is:

$$U(\epsilon) \equiv u(\tilde{c}_s^m) + \beta u(\tilde{c}_{s+1}^o) + V\left(\tilde{n}_s, \left(\frac{n_s \tilde{U}_{s+1} + \epsilon \tilde{U}_n(\epsilon)}{n_s + \epsilon}\right)\right)$$

[26] See Boldrin, De Nardi, and Jones (2005), p. 40, who discuss the failure of Ricardian equivalence in a similar context. However, they do not analyze efficiency.

Plugging in for the allocation $\tilde{z}$, the utility is:

$$U(\epsilon) \equiv u(c_s^m - \epsilon\theta) + \beta u(c_{s+1}^o + (\tau - b_{s+1})w_{s+1}\epsilon) + V\left(n_s + \epsilon, U_{s+1} - \left(\frac{u(\tilde{c}_n^m) - u(c_{s+1}^m)}{n_s + \epsilon}\right)\right)$$

Taking the derivative w.r.t. ϵ and evaluating at $\epsilon = 0$ and simplifying the expression becomes

$$\frac{\partial U(\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0} = -u'(c_s^m)\theta + \beta u'(c_{s+1}^o)(\tau - b_{s+1})w_{s+1} + V_n(n_s, U_{s+1}).$$

Note that from the FOCs we have:

$$V_n(n_s, U_{s+1}) = u'(c_s^m)\theta_t + \beta u'(c_{s+1}^o)b_{s+1}w_{s+1}.$$

Using the FOC to eliminate terms, we have $\frac{\partial U(\epsilon)}{\partial \epsilon}\big|_{\epsilon=0} = \beta u'(c_{s+1}^o)\tau$. This expression is positive if and only if $\tau > 0$.

Finally, no other generation is worse off. Hence, this allocation is $\mathcal{A}-$superior, which completes the proof. ■

This result is in contrast with the exogenous fertility dynastic OLG literature, starting with Barro (1974) and followed by Carmichael (1982), Burbidge (1983), Abel (1987) and others, where operative bequests or transfers are a sufficient condition for optimality or Pareto efficiency. The basic problem with a standard PAYG system is that the costs and benefits of producing children remain unaligned.

5.2 Fertility dependent PAYG pensions

The obvious way to align the cost and benefits of having children is to make the pension system fertility dependent (FDPAYG), the focus of this section.[27] Since parents are altruistic in our setup, FDPAYG also generates an increase in the desired transfer. If the FDPAYG system is large enough, the allocation of consumption levels is the same as in the case where parents have full property rights. Thus, FDPAYG can be used to implement an $\mathcal{A}-$efficient allocation. Interestingly, in the spirit of this result, several countries have now made provisions for time spent raising children to count towards

[27]Eckstein and Wolpin (1985), Abio, Mahieu, and Patxot (2004), Lang (2005) and Conde-Ruiz, Giménez, and Pérez-Nievas (2010) also point out that a fertility-dependent social security system is optimal. In contrast to our analysis, their results are derived in a model without altruism. Moreover, as mentioned before, the optimality concepts used differ from ours. Finally, property rights are assumed to lie with children throughout their analysis.

pension entitlements. In France, for example, a child supplement of 10% is added to social security benefits if the person raised at least three children.[28]

As before, the government taxes the middle aged at rate τ_t and gives the proceeds as a fertility dependent pension, $T_t(n_{t-1}) \equiv n_{t-1}\tau_t w_t$, to the old. That is, the parent knows that an increase in her own fertility affects her pension payment when old. Hence, the budget constraints now are:

$$\begin{aligned} c_t^m + \theta_t n_t + s_{t+1} &\leq w_t(1 + b_t - \tau_t) \\ c_{t+1}^o + b_{t+1} w_{t+1} n_t &\leq r_{t+1} s_{t+1} + T_{t+1}(n_t) \end{aligned}$$

Again, the FDPAYG system requires that the government balances its budget: $T_{t+1}(n_t) = n_t \tau_{t+1} w_{t+1}$.

To see why a large enough FDPAYG system leads to an $\mathcal{A}-$efficient allocation, consider the second budget constraint using the functional form for $T_t(n_{t-1})$:

$$c_{t+1}^o + (b_{t+1} - \tau_{t+1}) w_{t+1} n_t \leq r_{t+1} s_{t+1}$$

It is immediately apparent that private and government intergenerational transfers appear in exactly the same way. Therefore, whenever the transfer constraint is binding, by choosing a high enough tax rate, the government can undo the effect of the transfer constraint and therefore implement an $\mathcal{A}-$efficient allocation. The following proposition shows this formally.

Proposition 16 *If τ is large enough, the equilibrium allocation with FDPAYG is $\mathcal{A}-$efficient.*

Proof. We use the following change of variables. For all t, let $\tilde{b}_t = b_t - \tau_t$ and $\underline{\tilde{b}}_t = \underline{b}_t - \tau_t$. Then the household problem with FDPAYG is equivalent to maximizing (3) subject to

$$\begin{aligned} & c_t^m + \theta_t n_t + s_{t+1} \leq w_t(1 + \tilde{b}_t) \\ & c_{t+1}^o + \tilde{b}_{t+1} \leq r_{t+1} s_{t+1} \\ & \tilde{b}_{t+1} \geq \underline{\tilde{b}}_{t+1} \\ & c_t^m, c_{t+1}^o, n_t \geq 0 \end{aligned}$$

This is equivalent to the problem without FDPAYG. For all t, let b_t^* be the transfer chosen in a world without taxes and with $\underline{b}_t = -1$. By setting $\tau_t \geq \underline{b}_t - b_t^*$ for all t, we have

[28] Many other European countries have similar provisions, see Social Security Administration (2004).

$\underset{\sim}{\tilde{b}}_t \leq b_t^*$ for all t. Hence, the minimum transfer constraint above is not binding. Therefore, the equilibrium allocation is the same as the unconstrained equilibrium allocation without FDPAYG. By Proposition 6 this equilibrium allocation is $\mathcal{A}-$efficient. ∎

What happens here is that rather than parents taking from their own children, the government taxes all children and then allocates funds to the old taking the number of children into account.

Note that any FDPAYG system involving large enough transfers implements the *same* allocation. The parent will simply undo the government transfers by making larger private transfers. Thus, there is no unique "optimal tax," but an entire range of large enough FDPAYG taxes that implement the same $\mathcal{A}-$efficient allocation. Note that this result is different from Eckstein and Wolpin (1985), Abio, Mahieu, and Patxot (2004), Lang (2005) and Conde-Ruiz, Giménez, and Pérez-Nievas (2010), who all find a unique optimal fertility dependent tax level in related contexts but without altruism.[29]

This result speaks to the current policy debate that blames low fertility rates for the insolvency of the standard PAYG systems around the western world. While a social security system may have seemed like the obvious solution to old age poverty in a world where children were no longer obliged to look after their parents, it created inefficient distortions of fertility decisions.

5.3 Fertility subsidies and government debt

Another pronatalist policy that is seen to varying degrees in many countries are fertility subsidies. For example, many countries have tax deductions for children. Some countries also give a one-time subsidy for the birth of each child. For instance, the Russian government pays 4,500 Rubles (≈US$ 130) for the birth of each child. Similarly, several cantons in Switzerland and some cities in Italy pay large birth grants.[30] We now show that in the context of our model, fertility subsidies give an incentive to increase child-bearing and, if set at a high enough level, can lead to efficient fertility choices. In particular, we show that the unconstrained equilibrium allocation can be implemented through a policy that subsidizes fertility and finances these subsidies by issuing debt. The debt is then repaid by taxing the next generation, i.e., the children, in a lump-sum fashion a period later.

[29] Note that the allocation resulting from a large enough FDPAYG system is not the only efficient allocation. It is also not $\mathcal{A}-$superior to the allocation where parents are constrained and taxes are zero because, except for the initial old, every subsequent generation may be worse off.

[30] See Social Security Administration (2004) for details of such policies in European countries.

Let τ_t^s be the per child subsidy a parent receives and τ_t^d a labor income tax rate on all young people. Let d_{t+1} be per middle-aged person debt issued by the government.

$$c_t^m + \theta_t n_t + s_{t+1} + d_{t+1} \leq w_t(1 + b_t - \tau_t^d) + \tau_t^s n_t$$
$$c_{t+1}^o + b_{t+1} w_{t+1} n_t \leq r_{t+1}(s_{t+1} + d_{t+1})$$

Government budget balance (per old person) requires that

$$n_{t-1}(d_{t+1} + \tau_t^d w_t) = r_t d_t + \tau_t^s n_t n_{t-1}$$

holds in all periods.

Proposition 17 *Set $\tau_t^s = \tau_{t+1} \frac{w_{t+1}}{r_{t+1}}$ and set $\tau_t^d = \tau_t$ where τ_t are the taxes specified for the FDPAYG pension. Then, the equilibrium allocation with fertility subsidies and government debt is the same as under FDPAYG. Hence if τ_t^s and τ_t^d are large enough, the equilibrium allocation is $\mathcal{A}-$efficient.*

Proof. Combining the budget constraint when young and old by substituting out $(s_{t+1} + d_{t+1})$, it is straightforward to see that the household's budget set in period t with fertility subsidies and government debt (FSGD) is the same as for the FDPAYG pension. Hence, the chosen consumption, fertility and transfer allocation is the same in the two problems. Further, under FDPAYG, households receive $\tau_{t+1} n_t w_{t+1}$ when old while they receive $\tau_t^s n_t$ when young under FSGD. These are equal in present value. To achieve the same consumption allocation in the two periods, households have to save $\tau_t^s n_t$ more in FSGD than FDPAYG. If the government issues debt $d_{t+1} = \tau_t^s n_t$, then the government budget constraint holds every period and the government debt is exactly offset by the difference in private savings. Hence, the equilibrium capital stock does not change. Therefore, for large enough fertility subsidies, the same $\mathcal{A}-$efficient allocation can be implemented as an equilibrium outcome. ∎

In sum, fertility subsidies together with taxes on the next generation to finance these subsidies is identical, in our model, to allowing parents to leave negative bequests to their own children. In a more complicated model the two policies might not be exactly identical. In fact, fertility subsidies might be more desirable. For example in a world with uncertainty about the type (e.g., labor productivity) of one's own children, a fertility subsidy effectively offers insurance against low quality children. Such insurance is not offered by simply allowing parents to tap into their own children's income.

6 Conclusion

In this paper we analyze the effects of various degrees of parental control over children's labor income. We do this in the context of an OLG model with endogenous fertility where parents are altruistic towards children. We show that when parents don't have enough property rights, the costs and benefits of having children are not aligned, which leads to inefficiently low fertility. We show that the allocation of property rights also matters for more conventional efficiency concepts. For example, dynamic inefficiencies leading to the overaccumulation of capital are present only when people do not have enough property rights over their children. In addition, when fertility is endogenous, there is also potential underaccumulation of people. Property rights also matter for equilibrium efficiency in endogenous fertility models without altruism. Increasing the degree of altruism raises the threshold level of property rights beyond which these different types of inefficiencies occur.

We also show how property rights over children interact with other intergenerational policies. We show that a standard PAYG system will not lead to an $\mathcal{A}-$efficient allocation because even though taxes when middle-aged are lump-sum to children, they are distortionary for the parent and hence distort the fertility decision. We therefore examine alternative pension systems, in particular one where pension payments are a function of fertility choices, as well as fertility subsidies and government debt. Both systems are able to implement an $\mathcal{A}-$efficient allocation.

The paper points to several avenues for future work. First, it would be interesting to explore the positive implications of shifts in property rights over time. In particular, one would like to know to what extent historical changes in the allocation of property rights (from parents to children) have contributed to the demographic transition. This is a novel mechanism that hasn't been analyzed in the literature so far. While plausible, its historical relevance can only be assessed through a serious quantitative analysis.

A second positive avenue to pursue would be to analyze the importance of property rights for differential fertility. The set-up could be easily extended to to allow for heterogeneity. Constraints on transfers are likely to be binding only for some families. Introducing heterogeneity and analyzing the importance of legal changes for changes in differential fertility would be very interesting.[31]

Finally, in this paper, we take the shift in property rights as given and explore its consequences. Yet, a big open question is *why* laws shifting property rights from par-

[31]See De la Croix and Doepke (2003, 2004) for the importance of differential fertility for growth.

ents to children were introduced. At least two potential answers come to mind. One would be that legal constraints shifted for political economy reasons (e.g., that a majority of people voted for children's rights due to increased longevity, for example). Alternatively, the reason behind changes in de-facto ownership may have been driven by technological changes. For example, the change from an agricultural rural society to an industrialized urban society may have brought a change in the de facto control parents have over their children. We leave this investigation to future research.

A Appendix

A.1 Dynastic problem in aggregates

The constraint set in (2) is not convex in general. This is because n multiplies b in the budget constraint when old and both are choice variables. Therefore, the first-order conditions of this problem, while necessary, are not sufficient for an optimum. Instead of using second-order conditions to characterize the solution, one way to circumvent this problem is to follow Alvarez (1999) and to write the utility and constraints in terms of dynasty aggregates. This formulation is technically convenient in the proof of Proposition 14 and allows us to derive simple parameter conditions for the problem to be well defined when functional forms assumptions are made (see below).

That is, we can rewrite the utility of generation s as

$$U_s = \Omega\left(\left\{N^m_{s,t+1}\right\}^{\infty}_{t=s}, \left\{C^m_{s,t}, C^o_{s,t+1}\right\}^{\infty}_{t=s}\right) \tag{16}$$

where $N^m_{s,t} = \Pi^{t-1}_{k=s} n_k$ denotes the number of middle aged descendants of generation s in period t, $C^m_{s,t} = N^m_{s,t} c^m_t$ denotes total consumption of middle aged descendants of generation s in period t and $C^o_{s,t+1} = N^m_{s,t} c^o_{t+1}$ denotes total consumption of old descendants of generation s in period $t+1$. Similarly, the constraints can be rewritten as:

$$\begin{aligned}
&C^m_{s,t} + \theta_t N^m_{s,t+1} + S_{s,t+1} \leq N^m_{s,t} w_t + B_{s,t} w_t \\
&C^o_{s,t+1} + B_{s,t+1} w_{t+1} \leq r_{t+1} S_{s,t+1} \\
&B_{s,t+1} \geq N^m_{s,t+1} \underline{b}_{t+1} \\
&C^m_{s,t}, C^o_{s,t+1}, N^m_{s,t+1}, S_{s,t+1} \geq 0
\end{aligned} \tag{17}$$

where $S_{s,t+1} = N^m_{s,t} s_{t+1}$ and $B_{s,t} = N^m_{s,t} b_t$. Since the constraint set written in aggregates is convex, making the necessary monotonicity and concavity assumptions on Ω ensures that the first-order conditions in aggregates are sufficient for an optimum using standard results. In particular, we assume that to ensure that first-order conditions are necessary and sufficient we utility is bounded and strictly increasing and strictly concave in all its arguments, $\{N^m_{s,t+1}, C^m_{s,t}, C^o_{s,t+1}\}^{\infty}_{t=s}$.

A.1.1 Utility specifications in aggregates

For the utility specifications introduced in Section 3.3, the expressions in (12) can be expressed in aggregates as follows. Assuming $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$ and $g(x) = x^{1-\varepsilon}$, we get:

$$\begin{aligned} RB \qquad U_s &= \sum_{t=s}^{\infty} \zeta^t (N_{s,t}^m)^{\sigma-1} \left[\frac{(C_t^m)^{1-\sigma} + \beta (C_{t+1}^o)^{1-\sigma} + \gamma (N_{s,t+1}^m)^{1-\sigma}}{1-\sigma} \right] \\ BB \qquad U_s &= \sum_{t=s}^{\infty} \zeta^t (N_t^m)^{\sigma-\varepsilon} \left[\frac{(C_t^m)^{1-\sigma} + \beta (C_{t+1}^o)^{1-\sigma}}{1-\sigma} \right] \end{aligned} \tag{18}$$

which are the functions corresponding to Ω in equation 16.

Thus, the problem can be interpreted as the middle-aged adult in period s choosing $\left\{C_{s,t}^m, C_{s,t+1}^o, N_{s,t+1}^m, S_{s,t+1}, B_{s,t+1}\right\}_{t=s}^{\infty}$ to maximize U_s in equation (18) subject to the constraints in (17) for all $t \geq s$.

To ensure that utility is bounded, we assume

$$\zeta < 1. \tag{19}$$

Further, to ensure that first-order conditions are necessary and sufficient we assume that the utility is strictly increasing and strictly concave in all its arguments, $\{N_{s,t+1}^m, C_{s,t}^m, C_{s,t+1}^o\}_{t=s}^{\infty}$. Some of these conditions are useful in comparing the two specifications.

In particular, for utility to be strictly increasing in $N_{s,t}^m$ in the $RB-$altruism, the condition boils down to:

$$RB. \qquad \gamma u'(n_t) > \zeta \frac{\left[u'(c_{t+1}^m) c_{t+1}^m + \beta u'(c_{t+2}^o) c_{t+2}^o + \gamma u'(n_{t+1}) n_{t+1}\right]}{n_t}. \tag{20}$$

This condition says that the direct utility benefit has to be strictly larger than the indirect utility cost of diluting per generation consumption and fertility one period later.

With logarithmic utility, this condition boils down to:

$$RB(log). \qquad \gamma > \tfrac{\zeta(1+\beta)}{1-\zeta}. \tag{21}$$

Following Jones and Schoonbroodt (2010), there are three sets of joint parameter restrictions that ensure that utility satisfies the desired monotonicity and concavity properties for $BB-$type altruism:

$$\begin{aligned} &BB.1 \quad 0 < \varepsilon < \sigma < 1; \\ &BB.2 \quad 1 < \sigma < \varepsilon; \\ &BB.3 \quad 1-\varepsilon = \delta(1-\sigma) \text{ for some } \delta > 1 \text{ and } \sigma \to 1. \end{aligned} \tag{22}$$

In the last case, utility is separable and logarithmic and hence equivalent to the RB specification with logarithmic utility with $\gamma \equiv \frac{\delta\zeta(1+\beta)}{1-\zeta}$. Since $\delta > 1$, condition (21) is satisfied.[32]

A.1.2 First-order conditions in aggregates

Let Ω_x be the partial derivatives of Ω with respect to x. Let $\lambda_{B,t+1}$ be the Lagrange multiplier on the transfer floor in (17). The first-order conditions for the problem in aggregates are given by

$$\Omega_{C^m_{s,t}} = \Omega_{C^o_{s,t+1}} r_{t+1} \tag{23}$$

$$\Omega_{N^m_{s,t+1}} = \Omega_{C^m_{s,t}} \theta_t - \Omega_{C^o_{s,t+1}} w_{t+1} + \lambda_{B,t+1} \underline{b}_{t+1} \tag{24}$$

$$\Omega_{C^o_{s,t+1}} = \Omega_{C^m_{s,t+1}} + \frac{\lambda_{B,t+1}}{w_{t+1}} \tag{25}$$

A.2 Proofs of results in Section 4.2

A.2.1 Proof of Lemma 9

We closely follow the standard proof (see for example de la Croix and Michel (2002, Chapter 2). If $r < n$, the economy is in overaccumulation and aggregate output can be increased by saving less, holding population constant. Whether an increase in aggregate output translates into a Pareto improvement depends on the utility function. Our utility for generation s differs from the standard one in two ways. (1) Positive altruism: Earlier generations care about the utility of later generations. (2) Utility from fertility: Since fertility cannot be changed in a Pareto improvement, these terms enter as additive/multiplicative constants and can therefore be ignored. Therefore, an increase in aggregate output can always be translated into a Pareto improvement in this set up. Conversely, if $r \geq n$, then the economy is either at the golden rule or suffers from underaccumulation. Therefore, holding fertility constant, consumption cannot be

[32] Details on the necessary utility transformations that lead to this result are available upon request.

increased for some generation without decreasing it for another. Unlike the standard OLG model, altruism from parents to children implies that there may still be room for welfare improvement by decreasing one generation's consumption and increasing it for a later generation. If this was a welfare improvement for the early generation, they would have made higher transfers to the later generation in equilibrium. A contradiction.

A.2.2 Proof of Proposition 10

This result follows directly from equations (5) and (7), together with Propositions 6 and 7 that state that the equilibrium is inefficient if and only if the constraint is binding.

A.2.3 Proof of Proposition 11

To show that $r > n$ is necessary, note that the same allocation (with fixed population) that Pareto dominates a stationary equilibrium with $r < n$ in Lemma 9 also $\mathcal{A}-$dominates the equilibrium allocation.

To show that $r > n$ is not sufficient, note that at $\underline{b} = b^*$ we have $r > n$ by necessity. By continuity, there exists $\underline{b} > b^*$ such that $r > n$ in the resulting constrained equilibrium allocation. This together with Proposition 7 proves the result.

A.2.4 Proof of Proposition 14

Consider the first-order conditions in equations (23) to (25). If the equilibrium is unconstrained, then $\lambda_{B,t+1} = 0$. Substituting equations (23) and (25) into equation (24), we get:

$$\Omega_{N^m_{s,t+1}} = \Omega_{C^o_{s,t+1}}(r_{t+1}\theta_t - w_{t+1}).$$

Since $\Omega_{N^m_{s,t+1}} > 0$ by assumption, we have that $r_{t+1}\theta_t > w_{t+1}$ in an unconstrained equilibrium. By Proposition 6, it follows that $r_{t+1}\theta_t > w_{t+1}$ is necessary for $\mathcal{A}-$efficiency.

To show that the condition is not sufficient, realize that for $\underline{b} = b^*$ we have $r\theta > w$ by necessity. By continuity, there exists $\underline{b} > b^*$ such that $r\theta > w$. This together with Proposition 7 proves the result.

Note: Only in the special case where $\Omega_{N^m} = 0$, i.e. when the direct utility benefit and indirect utility cost of population cancel out, total population and aggregate capital are pure investment goods to produce aggregate consumption, and the rate of return condition requires $r\theta = w$ at any $\mathcal{A}-$efficient allocation.

A.3 Closed form solution for a special case

Here we explicitly derive a closed form solution for the special case of logarithmic utility together with a Cobb-Douglas production function, $F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}$, with $\alpha \in (0, 1)$.

First, suppose $\underline{b}_t = -1$ for all t. Then altruism implies that no generation is constrained. In this case, the steady state capital-labor ratio, fertility and transfers are given by:

$$k^* = \frac{\alpha\theta(\beta + \zeta(1 + \beta + \gamma))}{\beta(1 - \alpha) + \gamma - \alpha\zeta\,(1 + \gamma + \beta)} \tag{26}$$

$$n^* = \zeta A\alpha \left(\frac{\beta(1 - \alpha) + \gamma - \alpha\zeta\,(1 + \gamma + \beta)}{\alpha\theta(\beta + \zeta(1 + \beta + \gamma))} \right)^{1-\alpha} \tag{27}$$

$$b^* = \frac{[\zeta\theta\alpha(1 + \beta + \gamma) - (1 - \alpha)k^*\gamma]}{k^*(1 - \alpha)\,(\gamma - \zeta\,(1 + \gamma + \beta))} \tag{28}$$

Our parameter restriction (21) guarantees that all variables are strictly positive in equilibrium. Note that the optimal transfer may well be negative. We find that b^* is negative if and only if

$$\beta(1 - \alpha) > \alpha\zeta\,(1 + \gamma + \beta)\,. \tag{29}$$

To see this, note that b^* is negative if and only if

$$\theta\alpha\zeta(1 + \beta + \gamma) < (1 - \alpha)k^*\gamma.$$

Using equation (26) and rearranging yields condition (29). The condition is compatible with our parameter restriction (21) as long as $\zeta < \frac{\beta}{\alpha+\beta}$, i.e., as long as parents are not too altruistic.

Condition (29) shows that parents want to take resources from children if the labor share in output is sufficiently high and if parents value their children's utility little enough relative to their own old age consumption. This shows that even altruistic parents want to take resources away from their children under certain circumstances. It also suggests that children are not only a consumption good in this model, but also an investment good.

Second, consider $\underline{b}$ such that $b^* < \underline{b}$. In this case, the parent chooses $\hat{b} = \underline{b}$ and the steady state capital-labor ratio and fertility are given by:

$$\hat{k} = \frac{\alpha\beta\theta}{\alpha\gamma - (\beta+\gamma)(1-\alpha)\underline{b}} \quad (30)$$

$$\hat{n} = \frac{\gamma A\alpha(1-\alpha)\hat{k}^{\alpha}(1+\underline{b})}{(1+\beta+\gamma)\left(\alpha\theta + \underline{b}(1-\alpha)\hat{k}\right)} \quad (31)$$

For efficiency results in Section 4, it is useful to define the following two thresholds. Let b_P be the transfer constraint such that $\hat{n} = \hat{r}$ and let b_M be the transfer constraint such that $\hat{w} = \theta\hat{r}$. Using the equations above, we can derive closed form solutions for b_P and b_M:

$$b_P = \frac{\alpha(1+2\beta+\gamma) - \beta}{(1-\alpha)(1+2\beta+\gamma)} \quad (32)$$

$$b_M = \frac{\gamma\alpha - \beta(1-\alpha)}{(1-\alpha)(\beta+\gamma)} \quad (33)$$

Now, from the solution for $\hat{k}$, the maximal $\underline{b}$ for which a steady state equilibrium exists is $\underline{b}^{max} = \frac{\gamma\alpha}{(1-\alpha)(\beta+\gamma)}$. It is straightforward to see that $b_P < \underline{b}^{max}$ if and only if $(1+2\beta+\gamma)\alpha < 1$. Since this conditions does not contradict the parameter restrictions needed for the model to be well defined—conditions (21) and (19)—a low enough α is sufficient to guarantee the existence of b_P. Clearly, $b_M < \underline{b}^{max}$ is always true for admissible parameters.

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